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Flows about a Critical Point with a Single Zero Characteristic Root

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1. INTRODUCTION

We study the phase-portrait and invariant manifolds in a neighborhood of the origin for the system of differential equations

$$\dot{x} = Ax + X(x) \qquad (\cdot = d/dt), \tag{i}$$

where x is a real $(n + 1)$-vector. Here A is a real, constant, $(n + 1) \times (n + 1)$ matrix with one zero characteristic root and all remaining characteristic roots have real parts of the same sign, $X(x)$ is a real analytic vector function starting with terms of order greater than one, and $x = 0$ is an isolated critical point.

Such systems have been studied by Bendixson, Liapunov, Mendelson, and others. Bendixson [1, pp. 45–58] described the local phase-portrait in terms of sectors for the case $n = 1$. Using a reduction of (i) to a certain canonical form, Liapunov [5, p. 301] gave a rather complete description of the stability properties of $x = 0$ when $n \geqslant 1$. The first systematic study of the phase-portrait about $x = 0$ for $n > 1$ appears in Mendelson [6]. By imposing a certain condition on (i), Mendelson gave a partial description of the local phase-portrait in terms of "fan-cones" and "saddle-cones:" conical configurations in $(n + 1)$-space which are analogous to fans and saddles in the plane.

The literature contains many results which are related to system (i). For example, Dulac [2, pp. 358–366] considered complex systems of a more general type than (i); one consequence of his work is that (i) has a unique analytic stable (unstable) manifold when the nonzero characteristic roots of A all have negative (positive) real parts. That this stable (unstable) manifold is the only C^1 stable (unstable) manifold is implied by the work of Hartman [3, pp. 234–243]. A more recent concept than the stable manifold is that of a center manifold for a system which has characteristic roots with zero real parts. The existence of a C^m manifold for (i), for any positive integer m, is an immediate consequence of the work of Kelley [4]. Kelley also gives an example to show that, unlike the stable (unstable) manifold, a center manifold need not be unique.

In Section 3, we obtain a convenient canonical form for system (i), see Theorem 2. In particular, this canonical form satisfies the conditions which Mendelson [6] assumed in his investigation of the phase-portrait.

In Section 4 we prove that (i) has a C^∞ center manifold which is analytic except possibly at $x = 0$, see Theorem 3. Thus, in the case under consideration an improvement of Kelley's results is possible. We show by a means of a counterexample that this result cannot be further sharpened.

In Section 5 we define the notions of "fan-cylinder" and "saddle-cylinder," and give a complete description of the local phase-portrait of (i) in terms of them, see Theorem 5. This result completes the work of Mendelson, reduces to the theorem of Bendixson for $n = 1$, and implies the stability results of Liapunov. In addition, it provides a definite statement as to the number of center manifolds possessed by (i).

2. Preliminaries

Euclidean n-space will be denoted by E^n. The symbol $|\cdot|$ will denote the euclidean norm on vectors and the operator norm on matrices, and $\langle\cdot,\cdot\rangle$ will represent the standard inner product of a vector pair. The set of non-negative integers will be designated by N, and N^n will stand for the set of n-tuples of elements in N. A typical element of N^n will be denoted by $(m) = (m_1, \ldots, m_n)$; its norm will be written as $|m| = m_1 + \cdots + m_n$.

Let $x \in E^n$, and let $f = (f_1, \ldots, f_m)$ be a real m-vector valued function of x on an open subset U of E^n. We shall say that f is of class C^ω on U if and only if each of the coordinate functions f_i $(i = 1, \ldots, m)$ is real analytic on U; adding this to the standard notions the definition of a map of class C^r for any r in $\{0, 1, \ldots, \infty, \omega\}$ is obtained. Let f be of class C^r $(1 \leqslant r \leqslant \omega)$ on some open set U in E^n. The Jacobian matrix of f will be denoted by either f_x or $\partial_x f$. For any $(m) \in N^n$, the symbol $D_x^{(m)}$ will denote the operator

$$D_x^{(m)} = \partial^{|m|}/\partial x_1^{m_1} \cdots \partial x_n^{m_n}.$$

Suppose p is a real j-vector and q is a real k-vector such that $x = (p, q)$. Then the symbol $\partial_p f$ shall denote the $m \times j$ matrix $(\partial f_i/\partial p_l)$, $i = 1, \ldots, m$, $l = 1, \ldots, j$; $\partial_q f$ is defined similarly.

Let z be a complex l-vector, and let w be a complex m-vector. Let $j \in N$. The symbol $\{z, w\}_j^n$ will denote the set of all complex n-vector valued functions of (z, w) which have a convergent power series expansion in a neighborhood of (0, 0) beginning with terms of degree $\geqslant j$. The subset of $\{z, w\}_j^n$ consisting of all F whose power series expansions contain no terms in the components of z alone will be designated by $\{{}^*z, w\}_j^n$. Note that if $F \in \{{}^*z, w\}_j^n$, then $F(z, 0) \equiv 0$ in a neighborhood of $z = 0$. The subset $\{z, {}^*w\}_k^n$ is defined similarly. The subset of $\{z, w\}_j^n$ consisting of all F having power series expansions whose coefficients are real n-vectors will be represented by $[z, w]_j^n$.

A C^r-diffeomorphism R of an open neighborhood of the origin of E^n onto an open neighborhood of the origin in E^n which leaves 0 fixed will be called a local C^r-diffeomorphism on E^n. Let R be a local C^r-diffeomorphism on E^n and let f be a real valued function of class C^r on a neighborhood U of 0 in E^n which is never zero. Consider a real analytic system in E^n

$$\dot{x} = X(x) \qquad (\cdot = d/dt), \tag{ii}$$

where $X \in [x]_1^n$. The system obtained by multiplying X by f, namely $\dot{x} = f(x)\, X(x)$, has precisely the same phase-portrait as (ii) near the origin; the only difference between these two systems is the parametrizations of the orbits. We shall make extensive use of the following definition: Let $\dot{y} = Y(y)$ be a system obtained from (ii) by a multiplication by f followed by the change of variables $y = R(x)$. Then, we say that (ii) and $\dot{y} = Y$ are locally *C^r-equivalent*. It is clear that a local C^r-equivalence preserves the stability properties of the origin (except possibly for a change in direction) while locally invariant sets and critical points near the origin in each system are in one-to-one correspondence via the transformation $y = R(x)$.

We shall make use of the following two definitions: Let A be a complex, $n \times n$ matrix having Jordan form B. Let J be the variant of B obtained by substituting a nonzero real number η for 1, wherever 1 appears off the main diagonal. We shall say that J is in *Jordan η-form*. There exists a nonsingular, complex, $n \times n$ matrix P such that $P^{-1}AP = J$. If A is real, a nonsingular, real $n \times n$ matrix Q is obtained by using the real and imaginary parts of the columns of P in the usual way. The matrix $Q^{-1}AQ$ will be said to be in *real Jordan η-form.*

3. Canonical Forms

We consider a real analytic system in E^{n+1}

$$\dot{p} = Qp + P(p) \qquad (\cdot = d/dt), \tag{1}$$

where $P \in [p]_2^{n+1}$, Q is a real, constant, $(n+1) \times (n+1)$ matrix of rank n, the origin is a simple critical point (that is, the origin is a zero of P contained in an open neighborhood free of any other zeros), and the nonzero eigenvalues of Q all have real parts of the same sign.

Let η be an arbitrary nonzero real number, then (1) is locally C^ω-equivalent to the system

$$\dot{x}_0 = X_0(x_0, x), \qquad \dot{x} = Bx + X(x_0, x), \tag{2}$$

where: $x_0 \in E^1$, $x \in E^n$, B is a real, constant, $n \times n$ matrix, and

$$X_0 \in [x_0, x]_2^1, \qquad X \in [x_0, x]_2^n, \tag{3i}$$

The origin is a simple critical point, (3ii)

B is in real Jordan η-form, with eigenvalues $\nu_1, \ldots, \nu_n$ all having real parts of the same sign. (3iii)

By virtue of (3i, iii), system (2) has a (locally invariant) stable, or unstable C^ω manifold M of dimension n, according as $\operatorname{Re} \nu_i < 0$, or $\operatorname{Re} \nu_i > 0$ ($i = 1, \ldots, n$), respectively. The manifold M is locally unique, with respect to the class of C^1 manifolds. We shall use M to construct the next transformation. It follows from the results on p. 358 and succeeding remarks in Dulac [2] that the following theorem is true.

THEOREM 1. *There exists a unique $f \in [x]_2^1$ such that the change of variables given by $u_0 = x_0 - f(x)$, $u = x$ transforms (2) into*

$$\dot{u}_0 = U_0(u_0, u), \qquad \dot{u} = Bu + U(u_0, u)$$

where $U_0 \in [u_0, {}^*u]_2^1$, and $U \in [u_0, u]_2^n$.

As a consequence of Theorem 1, we may assume that (2) also satisfies the condition

$$X_0 \in [x_0, {}^*x]_2^1. \tag{4}$$

This means, of course, that the stable (unstable) manifold of (2) is (locally) the n-dimensional flat $x_0 = 0$.

The next transformation is based on a change of variables introduced by Liapunov [5, pp. 302–303]. Consider the equation

$$F(x_0, x) = Bx + X(x_0, x) = 0. \tag{5}$$

It follows from (3i, iii) and the implicit function theorem that (5) has a unique solution $x = \psi(x_0)$ such that $\psi \in [x_0]_1^n$. Differentiating (5) implicitly with respect to x_0, we obtain the sharper result

$$\psi \in [x_0]_2^n. \tag{6}$$

The map R: $u_0 = x_0$, $u = x - \psi(x_0)$ is therefore a local C^ω-diffeomorphism taking (2) into

$$\dot{u}_0 = U_0(u_0, u), \qquad \dot{u} = Bu + U(u_0, u), \tag{7}$$

where $U_0(u_0, u) = X_0(u_0, u + \psi(u_0))$, and

$$U(u_0, u) = B\psi(u_0) + X(u_0, u + \psi(u_0)) - U_0(u_0, u)\, \psi'(u_0). \tag{8}$$

In view of these definitions, (3i, iii), (4) and (6) imply that $U_0 \in [u_0, {}^*u]_2^1$ and $U \in [u_0, u]_2^n$.

The power series expansion of U_0 about (0, 0) must contain terms in u_0 alone. Otherwise, $U_0(u_0, 0) \equiv 0$ on an open interval I containing $u_0 = 0$; whence, by (8) and the definition of ψ, $U(u_0, 0) \equiv 0$ on I, so that (0, 0) is not a simple critical point of (7) which contradicts (3ii). Thus, there exists an integer $k \geqslant 2$ and a $U_1 \in [u_0]_0^1$ with $U_1(0) \neq 0$ such that

$$U_0(u_0, 0) = u_0{}^k U_1(u_0). \tag{9}$$

Hence, by definition of ψ and (8), $U(u_0, 0) \in [u_0]_{k+1}^n$.

It follows from (9) and the Weierstrass preparation theorem (in its real form) that U_0 has the factorization

$$U_0(u_0, u) = e(u_0, u)\, H_k(u_0, u),$$

where e is a real unit (i.e., $e \in [u_0, u]_0^1$, and $e(0, 0) \neq 0$), and H_k is a Weierstrass polynomial of degree k in u_0, of the form

$$H_k(u_0, u) = u_0{}^k + u_0^{k-1} U_{0,1}(u) + \cdots + u_0 U_{0,k-1}(u),$$

where $U_{0,j} \in [u]_1^1$ ($j = 1,..., k - 1$). The absence of terms in the components of u alone in H_k is a consequence of the fact that $U_0 \in [u_0, {}^*u]_2^1$.

Let $e^*(u_0, u) = [e(u_0, u)]^{-1}$, and let $e^*(0, 0) = c$. Set $A = cB$, $c\eta = \epsilon$, and $c\nu_i = \lambda_i$ ($i = 1,..., n$). Then, the matrix A is in Jordan ϵ-form and has eigenvalues $\lambda_1,..., \lambda_n$. We may assume ϵ is an arbitrary positive number. We now multiply $(U_0, Bu + U)$ by e^*. In view of the properties of (7), (1) is locally C^ω-equivalent to

$$\dot{x}_0 = x_0{}^k + X_0(x_0, x), \qquad \dot{x} = Ax + X(x_0, x), \tag{10}$$

where: $x_0 \in E^1$, $x \in E^n$, A is a real, constant, $n \times n$ matrix, and

$$X_0 \in [{}^*x_0, {}^*x]_2^1, \qquad X \in [x_0, x]_2^n, \tag{11i}$$

$$X(x_0, 0) \in [x_0]_{k+1}^n, \tag{11ii}$$

A is in real Jordan ϵ-form, with eigenvalues $\lambda_1,..., \lambda_n$ all having real parts of the same sign. (11iii)

Of course, (3ii) also holds, since this is true of all systems which are locally C^r-equivalent $(1 \leqslant r \leqslant \omega)$ to (1).

Using (10) we shall now obtain a canonical form of (1) which will be employed later to describe the local phase-portrait.

THEOREM 2. *System* (1) *is locally* C^ω*-equivalent to*

$$\dot{x}_0 = x_0{}^k + X_0(x_0, x), \qquad \dot{x} = Ax + X(x_0, x), \tag{12}$$

where:

$$X_0 \in [{}^*x_0, x]^1_{k+1}, \qquad X \in [x_0, x]^n_2, \tag{13i}$$

$$X(x_0, 0) \in [x_0]^n_{k+1}. \tag{13ii}$$

Proof. We may assume (1) has the form (10). Let C be a (complex) nonsingular, $n \times n$ matrix such that $C^{-1}AC = J$ is in Jordan ϵ-form. The change of variables $z_0 = x_0$, $z = C^{-1}x$ transforms (10) into

$$\dot{z}_0 = z_0{}^k + Z_0(z_0, z), \qquad \dot{z} = Jz + Z(z_0, z), \tag{14}$$

where $Z_0(z_0, z) = X_0(z_0, Cz)$ and $Z(z_0, z) = C^{-1}X(z_0, Cz)$; hence (11i, iii) imply that $Z_0 \in \{{}^*z_0, {}^*z\}^1_2$, $Z \in \{z_0, z\}^n_2$, and $Z(z_0, 0) \in \{z_0\}^n_{k+1}$.

We make the following general observation: If $h \in \{z_0, z\}^1_2$, then the map R: $w_0 = z_0 + h(z_0, z)$, $w = z$ transforms (14) into the system

$$\dot{w}_0 = w_0{}^k + G_h(z_0, z), \qquad \dot{w} = Jz + Z(z_0, z),$$

where

$$G_h(z_0, z) = z_0{}^k - (z_0 + h)^k + Z_0 + \partial_{z_0}h[z_0{}^k + Z_0] + \partial_z h[Jz + Z]. \tag{15}$$

We shall make use of the following lemma.

LEMMA 1. *There exists a unique polynomial* h *of degree* $\leqslant k$ *in* $\{{}^*z_0, z\}^1_2$ *such that* $G_h \in \{z_0, z\}^1_{k+1}$.

Proof. Let $(i) = (i_0, i_1, \ldots, i_n) \in N^{n+1}$, The vector (z_0, z) will be denoted by ζ, and $\zeta^{(i)}$ will denote the product $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$. Let $Z(\zeta) = (Z_1(\zeta), \ldots, Z_n(\zeta))$ and let

$$Z_j(\zeta) = \sum_{|i| \geqslant 2} Z^j_{(i)} \zeta^{(i)} \qquad (j = 0, \ldots, n).$$

The properties of (14) imply that

$$Z^j_{(i_0, 0, \ldots, 0)} = 0 \qquad (j = 0, \ldots, n; \quad i_0 = 2, \ldots, k). \tag{16}$$

We seek a polynomial $h = h(\zeta)$ of the form

$$h(\zeta) = \sum_{2 \leqslant |i| \leqslant k} h_{(i)} \zeta^{(i)},$$

such that $h \in \{{}^*z_0, z\}_2^1$, and $G_h \in \{z_0, z\}_{k+1}^1$. We shall use induction on i to show that these conditions uniquely determine the coefficients of h. This will prove Lemma 1.

We first rewrite (15) in the form

$$\sum_{l=1}^{n-1} \left(\lambda_l \frac{\partial h}{\partial z_l} + \epsilon_{l+1} \frac{\partial h}{\partial z_{l+1}} \right) z_l + \lambda_n z_n \frac{\partial h}{\partial z_n} = F(\zeta) + G_h(\zeta), \tag{17}$$

where

$$F(\zeta) = \sum_{l=1}^{k} \binom{k}{l} z_0^{k-l} h^l - \sum_{j=0}^{n} Z_j \frac{\partial h}{\partial z_j} - Z_0 - z_0^k \frac{\partial h}{\partial z_0} \tag{18}$$

and $\epsilon_j = 0$ or ϵ $(j = 2,..., n)$, depending on the precise form of J. Let δ_{ij} denote the Kronecker delta, and let $e_j = (\delta_{0j}, \delta_{1j}, ..., \delta_{nj})$. Let addition be defined in N^{n+1} in the usual way, and let

$$c_{(i)} = \sum_{j=1}^{n} i_j \lambda_j .$$

Then, in view of (17), the conditions which h has to satisfy are equivalent to

$$c_{(i)} h_{(i)} + \sum_{j=2}^{n} \epsilon_j (i_j + 1)\, h_{(i)+e_j-e_{j-1}} = F_{(i)}, \qquad (2 \leqslant |i| \leqslant k) \tag{19a}$$

$$h_{(i_0,0,\ldots,0)} = 0, \tag{19b}$$

where $F_{(i)}$ is the coefficient of $\zeta^{(i)}$ in $F(\zeta)$, and the following convention obtains: If $i_j < 0$ for any j, $h_{(i)} = 0$.

Since $(Z_0, Z) \in \{\zeta\}_2^{n+1}$, it follows from (18) that each $F_{(i)}$ $(|i| = 2,..., k)$ is a polynomial in $h_{(j)}$ with $|j| < |i|$ having no constant or linear terms. Thus, each $F_{(i)}$ is defined recursively. In particular, (16) and (18) imply that each $F_{(i_0,0,\ldots,0)}$ $(i_0 = 2,..., k)$ is a polynomial in $h_{(j_0,0,\ldots,0)}$ with $j_0 < i_0$, having no constant or linear terms. Therefore, (19a) and (19b) are compatible, and it remains only to determine $h_{(i)}$ for $(i) \neq (i_0, 0,..., 0)$.

We introduce the following ordering for the elements (i) such that $|i| = m$ $(m \geqslant 2)$: $(i) < (j)$ if and only if $i_l < j_l$ when l is the first integer such that $i_l \neq j_l$. Consider (19a) for all (i) such that $|i| = m$, and $(i) \neq (i_0, 0,..., 0)$, arranged in increasing order of (i) in accordance with $<$.

One obtains, in this way, a linear recursive system for the coefficients $h_{(i)}$ with $|i| = m$. Since $(i) + e_j - e_{j-1} < (i)$ for $j = 2,..., n$, the system is lower triangular with system determinant Δ_m given by

$$\Delta_m = \Pi\{c_{(i)}: |i| = m, \qquad (i) \neq (i_0, 0,..., 0)\}.$$

It follows from (11iii) that $\Delta_m \neq 0$ ($m = 2,..., k$); therefore, (19a) inductively determines the coefficients $h_{(i)}$, when $(i) \neq (i_0, 0,..., 0)$, in a unique fashion. This completes the proof of Lemma 1.

We now return to the proof of Theorem 2. Let h be as in Lemma 1, and set $f(x_0, x) = h(x_0, C^{-1}x)$. Clearly f is a polynomial of degree $\leqslant k$, and $f \in \{{}^*x_0, x\}_2^1$ (we recall that x_0 and x are real, but the coefficients may be complex). The map

$$R_2: u_0 = x_0 + f(x_0, x), \qquad u = x$$

transforms (11) into

$$\dot{u}_0 = u_0{}^k + H(x_0, x), \qquad \dot{u} = Ax + X(x_0, x),$$

where

$$H(x_0, x) = x_0{}^k - (x_0 + f)^k + X_0 + \partial_{x_0} f[x_0{}^k + X_0] + \partial_x f[Ax + X]. \quad (20)$$

It follows from the definition of f that $H \in \{x_0, x\}_{k+1}^1$.

As a matter of fact f has real coefficients, so that $f \in [{}^*x_0, x]_2^1$, and we may replace the braces by brackets in the preceding paragraph. One way to see this, is to note that the properties of f lead to a recursive system for the coefficients which is analogous to (19a), (19b), except that the system is real. Since this system is linear, it follows from the uniqueness of h that all the coefficients of f are real.

The foregoing implies that R_2 is a local C^ω-diffeomorphism. Let the inverse of R_2 be

$$R_2^{-1}: x_0 = u_0 + g(u_0, u), \qquad x = u,$$

where $g \in [u_0, u]_2^1$. We have, in fact, that $g \in [{}^*u_0, u]_2^1$, since $f \in [{}^*x_0, x]_2^1$. The map R_2 transforms (11) into

$$\dot{u}_0 = u_0{}^k + U_0(u_0, u), \qquad \dot{u} = Au + U(u_0, u), \quad (21)$$

where $U_0(u_0, u) = H(u_0 + g, u)$, and $U(u_0, u) = X(u_0 + g, u)$. Hence, using (11i, ii) and (20), we find that $U_0 \in [u_0, u]_{k+1}^1$, $U \in [u_0, u]_2^n$, and $U(u_0, 0) \in [u_0]_{k+1}^n$.

Since $U_0 \in [u_0, u]^1_{k+1}$, an application of the Weierstrass preparation theorem yields the factorization

$$u_0{}^k + U_0(u_0, u) = e(u_0, u)[u_0{}^k + \tilde{U}_0(u_0, u)],$$

where $e(u_0, u)$ is a unit such that $e(0, 0) = 1$, and $\tilde{U}_0 \in [{}^*u_0, u]^1_{k+1}$. Multiplying $(u_0{}^k + U_0, Au + U)$ by e^{-1} we obtain a system satisfying the hypotheses of Theorem 2. This completes the proof.

Remark 1. Unlike system (10), (12) need not, in general, satisfy the property that $x_0 = 0$ is the stable manifold. However, if (10) satisfies the additional condition

$$\begin{gathered} i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n \neq \lambda_j \qquad (j = 1,..., n), \\ \text{whenever } (i) = (i_1, ..., i_n) \in N^n \text{ is such that } |\, i \,| \geqslant 2, \end{gathered} \tag{*}$$

we may assume that $x_0 = 0$ is the stable manifold of (12). For, in this case we may use Sternberg's [7] Theorem 2 to linearize (10) on $x_0 = 0$; whereupon, it may be assumed that (10) satisfies $X \in [x_0, {}^*x]^n_2$. Then, it is easy to show that the stable manifold remains (locally) fixed under the map R_2.

4. The Center Manifold

We shall now analyze the center manifolds of system (1). The method we shall employ is analogous to that developed in Hartman [3, Chapter 9] for the study of the stable manifold. Any system which is locally C^ω-equivalent to (1) may be used to study the center manifolds. In particular, we shall assume that the system is given by (12).

A center manifold for (12) may be described in the following way: Let $g(x_0)$ be an n-dimensional, real, vector-valued function of class C^r $(1 \leqslant r \leqslant \omega)$ on $|\, x_0 \,| < \delta$ $(\delta > 0)$. The 1-dimensional, C^r manifold M^*, given by

$$M^* = \{(x_0, x): x = g(x_0), |\, x_0 \,| < \delta\},$$

is a C^r center manifold for (12) if and only if

$$g(0) = g'(0) = 0 \qquad (' = d/dx_0) \tag{22}$$

and the map

$$R: u_0 = x_0, u = x - g(x_0), \text{ with inverse } R^{-1}: x_0 = u_0, x = u + g(u_0), \tag{23}$$

transforms (12) into

$$\dot{u}_0 = U_0(u_0, u), \qquad \dot{u} = Au + U(u_0, u), \tag{24}$$

where

$$U(u_0, 0) \equiv 0 \qquad \text{on} \quad |u_0| < \delta. \tag{25}$$

It follows from (25) that $u = 0$ is locally invariant for (24) and, equivalently, M^* is locally invariant for (12).

We shall make use of the following equivalent characteristizations of a center manifold: Let $\xi = (x_0, x)$, and let $\phi(t, \xi)$ be the unique solution of (12) such that $\phi(0, \xi^0) = \xi^0$. Let $\{T^t: -\infty < t < \infty\}$ be the family of maps associated with (12), i.e., for each real t, $T^t: \xi^0 \to \xi^t$, where $\xi^t = \phi(t, \xi^0)$. (Note that the solutions may not exist for all time, so each T^t is defined at least on some set of points ξ^0 which contains $\xi^0 = 0$.) M^* is a C^r center manifold for (12) if and only if g satisfies (22), and M^* is locally invariant with respect to $\{T^t\}$. Using (12), (23) and (24), we find that

$$U(u_0, u) = Ag(u_0) + X(u_0, u + g) - [u_0^k + X_0(u_0, u + g)]\, g'(u_0).$$

Hence, by (25), M^* is a C^r center manifold for (12) if and only if g satisfies (22), and is a C^r solution of the vector differential equation

$$[x_0^k + X_0(x_0, g)]\, g' = Ag + X(x_0, g).$$

The main result of this section is the following theorem.

THEOREM 3. *There exists a* $\delta > 0$ *and a real n-dimensional vector-valued function* $g(x_0)$, *of class* C^∞ *on* $|x_0| < \delta$ *and of class* C^ω *on* $0 < |x_0| < \delta$, *such that*

$$g^{(m)}(0) = 0 \qquad (m = 0, 1, \ldots, k), \tag{26}$$

and the map (23) *transforms* (12) *into* (24), *where* (25) *holds. That is,* M^* *is a* C^∞ *center manifold for* (12) *which is analytic except possibly at the origin.*

Before proving Theorem 3, we shall obtain a number of preliminary lemmas.

First, we shall construct a system, depending on a parameter and of class C^∞ on E^{n+1}, which is locally C^ω-equivalent to (12). For each $\mu > 0$, the map

$$R_\mu: (y_0, y) = \mu^{-1}(x_0, x), \qquad \text{with} \quad R_\mu^{-1}: (x_0, x) = \mu(y_0, y),$$

is a local C^ω-diffeomorphism. Let $\zeta = (y_0, y)$. Note that ζ actually depends on μ, but this is tacit from the definition of R_μ, so we need not complicate the notation by indicating this dependence by, say, a subscript. Using R_μ, we see that (12) is locally C^ω-equivalent to

$$\dot{y}_0 = \tilde{Y}_0(\zeta, \mu), \qquad \dot{y} = Ay + \tilde{Y}(\zeta, \mu), \tag{27}$$

where $\tilde{Y}_0(\zeta, \mu) = \mu^{-1}[(\mu y_0)^k + X_0(\mu\zeta)]$ and $\tilde{Y}(\zeta, \mu) = \mu^{-1}X(\mu\zeta)$. If we select $\nu > 0$ such that $\{\zeta: |\zeta| \leqslant \nu\}$ is contained in the common domain of analyticity of X_0 and X, and we set $\mu_0 = \min\{1, \nu\}$, then (27) is analytic in ζ for $|\zeta| \leqslant 1$ and all μ in $(0, \mu_0]$. Let $\psi(s)$ be a real-valued, C^∞ function for $s \geqslant 0$ such that $\psi(s) = 1$ for $0 \leqslant s \leqslant 2^{-1}$, $0 < \psi(s) < 1$ for $2^{-1} < s < 1$, and $\psi(s) = 0$ for $s \geqslant 1$. Set

$$Y_0(\zeta, \mu) = \psi(|\zeta|^2)\, \tilde{Y}_0(\zeta, \mu) \text{ for } |\zeta| \leqslant 1, \text{ and } Y_0(\zeta, \mu) = 0 \text{ for } |\zeta| > 1,$$

and let $Y(\zeta, \mu)$ be defined analogously by replacing $\tilde{Y}_0$ with $\tilde{Y}$ in this formula. The system

$$\dot{y}_0 = Y_0(\zeta, \mu), \qquad \dot{y} = Ay + Y(\zeta, \mu) \tag{28}$$

is locally C^ω-equivalent to (12) via R_μ.

Let $\eta(t, \zeta^0, \mu)$ be the unique solution of (28) which satisfies $\eta(0, \zeta^0, \mu) = \zeta^0$. The map associated with $\eta(t, \zeta^0, \mu)$, namely

$$T_\mu{}^t: \zeta^0 \to \zeta^t = \eta(t, \zeta^0, \mu),$$

may be written in the form

$$T_\mu{}^t: y_0{}^t = y_0{}^0 + V_0(t, \zeta^0, \mu), \qquad y^t = e^{tA} y^0 + V(t, \zeta^0, \mu). \tag{29}$$

The properties of (29) are given in the following lemma; the proof which is an elementary extension of well known results (see Kelley [4, pp. 548–549] and Hartman [3, Lemma 3.1, p. 232]) will be omitted.

Lemma 2. *The map* (29) *satisfies the following properties:*

V_0, V are C^∞ for all (t, ζ^0, μ), such that $t \in (-\infty, \infty)$, $\zeta^0 \in E^{n+1}$, and $\mu \in (0, \mu_0)$; (30i)

$$V_0, V, \partial_{\zeta^0} V_0, \partial_{\zeta^0} V \equiv 0 \qquad \text{when} \quad \zeta^0 = 0; \tag{30ii}$$

There exists a positive number σ such that V_0, $V \equiv 0$ for $|\zeta^0| \geqslant \sigma$, $0 \leqslant t \leqslant 1$. (30iii)

For each $m \in N$, $(j) \in N^{n+1}$, and $\mu \in (0, \mu_0)$ define (30iv)

$$\beta(\mu, m) = \sum_{0 \leqslant |j| \leqslant m} \sup\{\max\{|D_{\zeta^0}^{(j)} V_0|, |D_{\zeta^0}^{(j)} V|\}: 0 \leqslant t \leqslant 1, \zeta^0 \in E^{n+1}\}.$$

Then, for each m in N there exists a nondecreasing, bounded function $\theta_m(\mu)$ on $(0, \mu_0)$ such that $\beta(\mu, m) \leqslant \theta_m(\mu)$ for each μ in $(0, \mu_0)$, and $\theta_m(\mu) \to 0$ as $\mu \to 0$.

We shall now obtain a center manifold for (28), whenever μ is sufficiently small, using the group of maps $\{T_\mu{}^t\}$. First we shall obtain a suitable invariant manifold for the map $T_\mu = T_\mu{}^1$; then we show that this manifold is invariant with respect to the full group $\{T_\mu{}^t\}$. To simplify the notation, we shall denote $(y_0{}^0, y^0)$ by (y_0, y). Setting $t = 1$ in (29), and deleting the constant 1 in the arguments of V_0 and V, we obtain

$$T_\mu\colon y_0{}^1 = y_0 + V_0(\zeta; \mu), \qquad y^1 = Cy + V(\zeta; \mu), \tag{31}$$

where $C = e^A$.

By virtue of (11iii), we may assume that each eigenvalue of A has positive real part. Therefore, we may choose $\epsilon > 0$ (in the Jordan ϵ-form of A) such that if we set $| C^{-1} | = c^{-1}$, then

$$| C^{-1} |^{-1} = c > 1. \tag{32}$$

In what follows, the symbol $\|f\|$ will denote the supremum norm of a vector function f which is bounded on E^1.

Lemma 3. *There exists a number a, $0 < a < \mu_0$, such that for each $\mu \in (0, a]$ there exists a unique n-dimensional real vector-valued function $f_\mu(y_0)$ satisfying the following conditions: f_μ is of class C^1 on E^1, of class C^∞ at $y_0 = 0$, $\|f_\mu{}'\| \leqslant 1$ $({}' = d/dy_0)$,*

$$f_\mu(0) = f_\mu{}'(0) = 0, \tag{33}$$

and the map

$$S_\mu\colon u_0 = y_0, \qquad u = y - f_\mu(y_0) \tag{34}$$

transforms T_μ into

$$S_\mu T_\mu S_\mu^{-1}\colon u_0{}^1 = u_0 + W_0(u_0, u; \mu),\ u^1 = Cu + W(u_0, u; \mu), \tag{35}$$

where W_0, W, $\partial_{(u_0,u)} W_0$, $\partial_{(u_0,u)} W \equiv 0$ when $(u_0, u) = (0, 0)$, and

$$W(u_0, 0; \mu) \equiv 0. \tag{36}$$

The proof of Lemma 3 will be deferred until we obtain some preliminary results. We note first that by virtue of (31), (34), and (35),

$$W(u_0, u; \mu) = Cf_\mu + V(u_0, u + f_\mu; \mu) - f(\mu_0 + V_0(u_0, u + f_\mu; \mu)).$$

Therefore, f satisfies the existence assertion of Lemma 3 if and only if f_μ is of class C^1 on E^1, of class C^∞ at $y_0 = 0$, $\|f_\mu{}'\| \leqslant 1$, and satisfies (33) together with

$$f_\mu = C^{-1}\{f_\mu(y_0 + V_0(y_0, f_\mu; \mu)) - V(y_0, f_\mu; \mu)\}. \tag{37}$$

Let σ be as in (30iii). If f_μ is a C^1 solution of (37) then $f_\mu(y_0) \equiv 0$ for $| y_0 | \geqslant \sigma$, and $| f_\mu(y_0)| < \sigma$ for all $y_0 \in E^1$. To see this, we note that by virtue of (30iii) and (37), $f_\mu(y_0) = C^{-1}f_\mu(y_0)$ whenever either $|y_0| \geqslant \sigma$, or $|f_\mu(y_0)| \geqslant \sigma$; whence, the assertion follows from (32).

For any positive integer m, define

$$\mathscr{F}_m = \{f = f(s) \text{ satisfying (38i–iii)}\}$$

where

$$f: E^1 \to E^n \text{ is of class } C^m \text{ on } E^1. \tag{38i}$$

$$f(0) = f'(0) = 0 \qquad (' = d/ds). \tag{38ii}$$

$$f(s) \equiv 0 \qquad \text{when} \quad |s| \geqslant \sigma. \tag{38iii}$$

Set

$$h_\mu(s, f) = s + V_0(s, f; \mu), \tag{39}$$

and define the operator Φ_μ by

$$\Phi_\mu[f](s) = C^{-1}[f(h_\mu(s, f)) - V(s, f(s); \mu)]. \tag{40}$$

Then, in view of (30i–iii), $\Phi_\mu: \mathscr{F}_m \to \mathscr{F}_m$ for each $\mu \in (0, \mu_0)$. We now see that Lemma 3 is equivalent to the following statement: $\Phi_\mu: \mathscr{F}_1 \to \mathscr{F}_1$ has a unique fixed point f_μ, which is C^∞ at $s = 0$, satisfying $\|f_\mu'\| \leqslant 1$, whenever $\mu \in (0, a]$.

LEMMA 4. *There exists a μ_* in $(0, \mu_0)$ such that whenever $\mu \in (0, \mu_*]$, Φ_μ has at most one fixed point $f \in \mathscr{F}_1$, satisfying $\|f_\mu'\| \leqslant 1$.*

Proof. Let f_1 and f_2 both satisfy the above hypotheses. It is easy to show, using (30iv) and the mean value theorem, that

$$|f_1(s) - f_2(s)| \leqslant c^{-1}[|f_1(h_\mu(s, f_1)) - f_2(h_\mu(s, f_1))| + 2\theta_1(\mu)\,|f_1(s) - f_2(s)|].$$

Hence, $\|f_1 - f_2\| \leqslant c^{-1}(1 + 2\theta_1(\mu))\,\|f_1 - f_2\|$. Thus, in view of (30iv) and (32) we can select $\mu_* \in (0, \mu_0)$ such that

$$\|f_1 - f_2\| \leqslant K\|f_1 - f_2\| \qquad \text{for} \quad 0 < \mu \leqslant \mu_*,$$

where $K < 1$. Whence, the proof of Lemma 4 follows.

For $f \in \mathscr{F}_m$, define $f^{(r)}$ as the rth derivative of f ($f^{(0)} = f$), and let

$$\|f\|_m = \max\{\|f^{(r)}\|: 1 \leqslant r \leqslant m\}.$$

We shall now obtain estimates for the derivatives $\Phi_\mu[f]^{(r)}$.

LEMMA 5. *Suppose $f \in \mathscr{F}_m$. Then, for each $1 \leqslant r \leqslant m$*

$$\|\Phi_\mu[f]^{(r)}\| \leqslant c^{-1}[P_r(\|f'\|, \mu)\,\|f\|_r + Q_r(\|f\|_{r-1}, \mu)], \tag{41}$$

where $P_r(t, \mu)$ *and* $Q_r(t, \mu)$ *are polynomials in* t *with coefficients which are nonnegative, nondecreasing functions of* μ *on* $(0, \mu_*]$ *such that* $P_r(t, \mu) \to 1$ *and* $Q_r(t, \mu) \to 0$ *as* $\mu \to 0$ *for each real* t. *In particular,* $Q_1 = \theta_1(\mu)$ *as given by* (30iv).

Proof. Let $(r, 0)$ denote $(r, 0,..., 0) \in N^{n+1}$, and let $f = (f_1,..., f_n)$. Differentiating (40), we obtain

$$\Phi_\mu[f]^{(1)}(s) = C^{-1}\{[h_\mu'(s, f) f'[h_\mu(s, f)] - \partial_{y_0} V(s, f; \mu) \\ -\partial_y V(s, f; \mu) f'(s)\}, \tag{42.1}$$

and, in general, for any $1 < r \leqslant m$ (if $m > 1$),

$$\begin{aligned}\Phi_\mu[f]^{(r)}(s) = C^{-1}\{&[h_\mu'(s, f)]^r f^{(r)}[h_\mu(s, f)] \\ &+ h_\mu^{(r)}(s, f) f'[h_\mu(s, f)] - \partial_y V(s, f; \mu) f^{(r)}(s) \\ &- D_\zeta^{(r,0)} V(s, f; \mu) + \sum_{l=2}^{r-1} G_l f^{(l)}[h_\mu(s, f)] \\ &- \sum_{1 \leqslant |i| \leqslant r} H_{(i)} D_\zeta^{(i)} V(s, f; \mu)\},\end{aligned} \tag{42.r}$$

where each G_l $(2 \leqslant l \leqslant r - 1)$ is a finite sum of terms of the form

$$h_\mu^{(r_1)}(s, f)\, h_\mu^{(r_2)}(s, f) \cdots h_\mu^{(r_q)}(s, f),$$

where $0 < r_j < l$ $(j = 1,..., q)$, and each $H_{(i)}$ $(1 \leqslant |i| \leqslant p)$ is a finite sum of terms of the type

$$[f_1^{(r_1)}(s)]^{j_1}[f_2^{(r_2)}(s)]^{j_2} \cdots [f_n^{(r_n)}(s)]^{j_n},$$

where $j_q \in N$, $0 < r_q < r$ $(q = 1,..., n)$, and $j_1 + j_2 + \cdots + j_n = |i|$.

We note that

$$h_\mu'(s, f) = 1 + \partial_{y_0} V_0(s, f; \mu) + \partial_y V_0(s, f; \mu) f'(s),$$

while for each l such that $1 < l \leqslant r$,

$$h_\mu^{(l)}(s; f) = D_\zeta^{(l,0)} V_0(s, f; \mu) + \partial_y V_0(s, f; \mu) f^{(l)}(s) \\ + \sum_{1 \leqslant |i| \leqslant l} L_{(i)} D_\zeta^{(i)} V_0(s, f; \mu),$$

where each $L_{(i)}$ $(1 \leqslant |i| \leqslant l)$ is a finite sum of terms of the form

$$[f_1^{(l_1)}(s)]^{j_1}[f_2^{(l_2)}(s)]^{j_2} \cdots [f_n^{(l_n)}(s)]^{j_n},$$

where $0 < l_q < l$ $(q = 1,..., n)$, and j_q is as previously defined.

We now rewrite (42.r), for $1 < r \leqslant m$, as

$$\begin{aligned} \Phi_\mu[f]^{(r)}(s) = C^{-1}\{&[h_\mu'(s, f)]^r f^{(r)}(s) \\ &+ [\langle \partial_y V_0(s, f; \mu), f^{(r)}(s)\rangle f'(s) \\ &- \partial_y V(s, f; \mu) f^{(r)}(s)] + M_r(s, f; \mu)\}, \end{aligned} \tag{43.r}$$

where

$$\begin{aligned} M_r = &- D_\zeta^{(r,0)} V(s, f; \mu) + [D_\zeta^{(r,0)} V_0(s, f; \mu) \\ &+ \sum_{1\leqslant|i|\leqslant r} L_{(i)} D_\zeta^{(i)} V_0(s, f; \mu)] f'[h_\mu(s, f)] \\ &+ \sum_{l=2}^{r-1} G_l f^{(l)}[h_\mu(s, f)] - \sum_{1\leqslant|i|\leqslant r} H_{(i)} D_\zeta^{(i)} V(s, f; \mu). \end{aligned}$$

The definitions of $H_{(i)}$, $L_{(i)}$ and $\|f\|$ imply that there exists a constant $K > 0$ such that $|H_{(i)}|$, $|L_{(i)}| \leqslant K(\|f\|_{r-1})^r$ for all (i) such that $1 \leqslant |i| \leqslant r$. Hence, by virtue of (30iv), there exists a constant $K_1 > 0$ such that

$$|h_\mu^{(l)}| \leqslant \theta_r(\mu)[1 + \|f\|_l + K_1(\|f\|_{r-1})^r]$$

for each $l = 1,..., r - 1$. Denote this majorant by Λ_l. Then the definition of G_l implies that there exists a constant $K_2 > 0$ such that $|G_l| \leqslant K_2\Lambda_{r-1}^r$ for each $l = 1,..., r - 1$.

In view of the definition of M_r, the above paragraph implies the existence of a function $Q_r(t, \mu)$ satisfying the hypotheses of Lemma 5 such that $|M_r| \leqslant Q_r(\|f\|_{r-1}\mu)$ for $r > 1$. Consequently, using (30iv) in a simple majorizing calculation for (43.r) $[r = 1,..., m]$, we obtain (41), where Q_r is as defined above for $r > 1$, $Q_1(t, \mu) = \theta_1(\mu)$, $P_1(t, \mu) = 1 + (2 + t)\,\theta_1(\mu)$ and $P_r(t, \mu) = [1 + (1 + t)\,\theta_1(\mu)]^r + (1 + t)\,\theta_1(\mu)$ for $r > 1$. This completes the proof of Lemma 5.

For each positive integer m, let

$$\mathscr{G}_m = \{f \in \mathscr{F}_m: \|f\|_m \leqslant \tfrac{1}{2}\}.$$

(There is nothing special about $\frac{1}{2}$ here; any positive number less than one would do equally as well in what follows.) In view of (32), an immediate corollary of Lemma 5 is the following result.

LEMMA 6. *There exists a nonincreasing sequence of positive numbers* $\{\mu_m\}$, *with* $0 < \mu_1 \leqslant \mu_*$, *such that* $\Phi_\mu(\mathscr{G}_m) \subset \mathscr{G}_m$ *whenever* $\mu \in (0, \mu_m]$.

Let m be a positive integer, $m \geqslant 2$. Select a function $f_{0,\mu} \in \mathscr{G}_m$, say $f_0 = 0$, and consider for each $\mu \in (0, \mu_*]$ the sequence of successive approximations

$$f_{n+1,\mu} = \Phi_\mu[f_{n,\mu}], \qquad n \in N. \tag{44}$$

In view of Lemma 6, the usual type of argument employing the Arzelà–Ascoli theorem implies the existence of $f_\mu \in \mathscr{G}_{m-1}$ such that $f_\mu = \Phi_\mu[f_\mu]$, for each $\mu \in (0, \mu_m]$. Moreover, by Lemma 4, f_μ is the only such function in $\mathscr{F}_1$ satisfying $\|f_\mu'\| \leqslant 1$. We have, therefore, proved the following lemma.

LEMMA 7. *Let* μ_m, $m \geqslant 2$, *be as in Lemma* 6. *Then, for each* $\mu \in (0, \mu_m]$, *there exists an* $f_\mu \in \mathscr{G}_{m-1}$ *such that*

$$f_\mu = \Phi_\mu[f_\mu].$$

Furthermore, f_μ *is the only fixed point of* Φ_μ *in* $\mathscr{F}_1$ *which satisfies* $\|f_\mu'\| \leqslant 1$.

We are now ready to give the proof of Lemma 3.

Proof of Lemma 3. Let $\{\mu_m\}$ be as in Lemma 6, and let f_μ be as in Lemma 7. It follows from the discussion after (40) that it only remains to find $a \in (0, \mu_2]$ such that f_μ is of class C^∞ at $s = 0$ whenever $\mu \in (0, a]$.

Let $\tilde{\mu} = \inf\{\mu_m : m \geqslant 2\}$. Let $a = \tilde{\mu}$ if $\tilde{\mu} > 0$, and $a = \mu_3$ if $\tilde{\mu} = 0$. Suppose first that $\tilde{\mu} > 0$, and let $\mu \in (0, a]$. Then, $\mu \in (0, \mu_m]$ for all $m \geqslant 2$, and it, therefore, follows that $f_\mu \in \mathscr{G}_m$ for all $m \geqslant 2$. Thus, Lemma 3 holds in this case.

Suppose next that $\tilde{\mu} = 0$ so that $a = \mu_3$. Let $\mu \in (0, a]$. Then, there exists an integer $m_0 \geqslant 3$ such that $\mu_{m_0+1} < \mu \leqslant \mu_{m_0}$. Hence, $f_\mu \in \mathscr{G}_{m_0-1} \subset \mathscr{G}_2$. To complete the proof, it suffices to show that for any $m \geqslant m_0 + 1$ there exists a $\delta_m > 0$ such that f_μ is of class C^{m-1} on $|s| < \delta_m$. We shall accomplish this by using the map R_μ introduced earlier.

Let $m \geqslant m_0 + 1$. Since the sequence $\{\mu_m\}$ is nonincreasing, $\mu_m < \mu$. Denote the system obtained from (28) by fixing the parameter at μ, and μ_m by (28_μ), and (28_{μ_m}), respectively. Recall that $R_\mu[R_{\mu_m}]$ is a local C^ω-diffeomorphism of (12) into (28_μ) $[(28_{\mu_m})]$. Therefore, the map $R_{\mu_m}R_\mu^{-1}$ is a local C^ω-diffeomorphism of (28_μ) into (28_{μ_m}). Hence,

$$(R_{\mu_m}R_\mu^{-1})\, T_\mu{}^t(R_\mu R_{\mu_m}^{-1}) = T_{\mu_m}^t$$

on some appropriate η-neighborhood of the origin. Thus, on setting $t = 1$, we see from (29) that

$$(V_0(\zeta; \mu_m), V(\zeta; \mu_m)) = \mu\mu_m^{-1}(V_0(\mu^{-1}\mu_m\zeta; \mu), V(\mu^{-1}\mu_m\zeta; \mu)) \tag{45}$$

for $|\zeta| < \eta$.

Now, we define

$$\hat{f}(s) = \mu\mu_m^{-1} f_\mu(\mu^{-1}\mu_m s), \tag{46}$$

and select $\delta' > 0$ such that $|(s, \hat{f}(s))| < \eta$, whenever $|s| < \delta'$. Using the fact that $f_\mu \in \mathscr{G}_2$ and $\mu^{-1}\mu_m < 1$, we see that $\hat{f}$ satisfies all the properties of $\mathscr{G}_2$ except possibly (38iii): $\hat{f}(s) \equiv 0$ for $|s| \geqslant \mu\mu_m^{-1}\sigma$, and not necessarily for $|s| \geqslant \sigma$. It follows from (40), (45) and (46) that

$$\Phi_{\mu_m}[\hat{f}](s) = \mu\mu_m^{-1}\Phi_\mu[f_\mu](\mu^{-1}\mu_m s) \qquad \text{whenever} \quad |s| < \delta',$$

and, therefore, since $f_\mu = \Phi_\mu[f_\mu]$ it follows from (46) that

$$\hat{f}(s) = \Phi_{\mu_m}[\hat{f}](s) \qquad \text{for} \quad |s| < \delta'. \tag{47}$$

Define the sequence $\{\hat{f}_{n,\mu_m}\}$ by $\hat{f}_{0,\mu_m} = \hat{f}$, and $\hat{f}_{n,\mu_m} = \Phi_{\mu_m}[\hat{f}_{n-1,\mu_m}]$ for $n \geqslant 1$. Although $\hat{f}_{0,\mu_m}$ need not belong to $\mathscr{G}_2$, $\hat{f}_{n,\mu_m} \in \mathscr{G}_2$ for all $n \geqslant 1$. Using the proof of Lemma 7, we obtain a function $\hat{f}_{\mu_m} \in \mathscr{G}_1$ such that

$$\|\hat{f}_{n,\mu_m} - \hat{f}_{\mu_m}\| \to 0 \qquad \text{as} \quad n \to \infty \quad \text{and} \quad \hat{f}_{\mu_m} = \Phi_{\mu_m}[\hat{f}_{\mu_m}].$$

Therefore, by Lemma 7, $\hat{f}_{\mu_m} = f_{\mu_m}$. Hence, (47) and the definition of $\hat{f}_{\mu_m}$ imply that

$$\hat{f}(s) = f_{\mu_m}(s) \qquad \text{for} \quad |s| < \delta'.$$

Consequently, setting $\delta_m = \mu^{-1}\mu_m\delta'$, we have from (46) that

$$f_\mu(s) = \mu^{-1}\mu_m f_{\mu_m}(\mu\mu_m^{-1}s) \qquad \text{whenever} \quad |s| < \delta_m. \tag{48}$$

Clearly f_μ is of class C^{m-1} on $|s| < \delta_m$. This completes the proof of Lemma 3.

It follows from the above work that for each $\mu \in (0, a]$,

$$M_\mu{}^* = \{\zeta : y = f_\mu(y_0)\}$$

is a 1-dimensional, invariant, C^2 manifold for $T_\mu = T_\mu{}^1$ which is of class C^∞ at the origin. We shall now show that $M_\mu{}^*$ is invariant with respect to $\{T_\mu{}^t\}$ if μ is sufficiently small.

Since f_μ given by Lemma 3 is a member of $\mathscr{G}_1$, we have in particular, $\|f_\mu'\| \leqslant 2^{-1}$. By Lemma 2, we can choose $a_0 \in (0, a]$ such that $\theta_1(\mu) < 2^{-1}$ for $0 < \mu \leqslant a_0$. With these definitions, an obvious modification of the proof of Corollary 5.1, of Hartman [3, p. 238] produces the following result.

LEMMA 8. *Let T_μ, f_μ, c and a_0 be as given previously. Let $\mu \in (0, a_0]$. For* *a given* $\zeta = (y_0, y) \in E^{n+1}$ *put* $\zeta^n = T_\mu{}^n(\zeta)$, $n = 0, \pm 1, \pm 2, \ldots$, *where* $\zeta^0 = \zeta$. *Then,*

$$|\zeta^n| = 0[(1 + 2\theta_1(\mu))^n] \qquad \textit{as} \quad n \to \infty, \quad \textit{if} \quad \zeta \in M_\mu{}^*,$$

and

$$0(|\zeta^n|) = [c - 2\theta_1(\mu)]^n \qquad \textit{as} \quad n \to \infty \quad \textit{when} \quad \zeta \notin M_\mu{}^*.$$

It follows from (30iv) and (32) that there exist numbers $a_* \in (0, a_0]$ and $b \in (0, 1)$ such that

$$1 + 2\theta_1(\mu) \leqslant b[c - 2\theta_1(\mu)] \qquad \text{whenever} \quad \mu \in (0, a_*].$$

Therefore, in view of Lemma 8, if $\zeta \in M_\mu{}^*$ and $\zeta_1 \notin M_\mu{}^*$, then

$$\lim_{n\to\infty} (|\zeta^n|/|\zeta_1{}^n|) = 0, \qquad \text{for each} \quad 0 < \mu \leqslant a_*.$$

A straightforward modification of the proof of Hartman's [3, p. 240] Corollary 5.2 gives the following lemma.

LEMMA 9. *Let* T_μ, f_μ, *and* a_* *be as discussed previously. The map* (34) *transforms* $\{T_\mu{}^t\}$ *into the group* $\{S_\mu T_\mu{}^t S_\mu^{-1}\}$, *which we represent by*

$$S_\mu T_\mu{}^t S_\mu^{-1}\colon u_0{}^t + W_0(t, u_0, u, \mu), \qquad u^t = e^{tA}u + W(t, u_0, u, \mu).$$

Then, for $-\infty < t < \infty$, $u_0 \in E^1$ *and* $\mu \in (0, a_*]$ *we have*

$$W(t, u_0, 0, \mu) \equiv 0;$$

which means that $M_\mu{}^*$ *is invariant with respect to* $\{T_\mu{}^t\}$ *or, equivalently,* $M_\mu{}^*$ *is a center manifold for system* (28).

Since system (28) is locally C^ω-equivalent to (12) via R_μ, it follows from Lemma 9 that for each $\mu \in (0, a_*]$,

$$R_\mu^{-1}(M_\mu{}^*) = \{\zeta\colon x = \mu f_\mu(\mu^{-1}x_0)\}$$

is a 1-dimensional, locally invariant, C^2 manifold for (12) which is C^∞ at the origin. Actually each of these manifolds is locally identical. To see this let $\mu_1, \mu_2 \in (0, a_*]$, where $\mu_1 < \mu_2$. Then, employing the same argument used to obtain (48), we find an $\eta > 0$ such that

$$f_{\mu_2}(x_0) = \mu_2^{-1}\mu_1 f_{\mu_1}(\mu_2\mu_1^{-1}x_0), \qquad \text{whenever} \quad |x_0| < \eta.$$

Therefore, $\mu_2 f_{\mu_2}(\mu_2^{-1}x_0) = \mu_1 f_{\mu_1}(\mu_1^{-1}x_0)$ *for* $|x_0| < \mu_2\eta$.

This proves our assertion. Since we are interested only in the local properties of (12), we shall take $\mu f_\mu(\mu^{-1}x_0)$ to be independent of μ.

We shall now prove the main result of this section.

Proof of Theorem 3. Let f_μ and a_* be defined as before. Select any $\mu \in (0, a_*]$. Now set $g(x_0) = \mu f_\mu(\mu^{-1}x_0)$ on $| x_0 | < \delta$, where $\delta > 0$ is such that R_μ defines a local C^ω-diffeomorphism of (12) into (28) on $| x_0 |, | x | < \delta$. Define

$$M^* = \{\xi: x = g(x_0) \text{ and } | x_0 | < \delta\}. \tag{49}$$

We may also assume, in light of the properties of (12), that δ is such that M^* is contained in an open neighborhood of $\xi = 0$ in which (12) is analytic and has no critical points save the origin.

Let R be given by (23). By virtue of Lemmas 3 and 9, it remains only to prove that g is C^ω on $0 < |x_0| < \delta$ and satisfies (26) for $2 \leqslant m \leqslant k$.

The analyticity of g on $0 < | x_0 | < \delta$ is an easy consequence of the fact that M^* is an invariant curve for the analytic system (12), and the definition of δ.

Property (26) holds for $m = 0, 1$. Assume that $g^{(l)}(0) = 0$ for all l such that $0 \leqslant l < m \leqslant k$. Since M^* is a center manifold, we recall that

$$g(x_0) \equiv A^{-1}\{[x_0^k + X_0(x_0, g(x_0))]\, g'(x_0) - X(x_0, g(x_0))\}$$

on $| x_0 | < \delta$. Differentiating this identity m times and setting $x_0 = 0$, it follows easily from (13i, ii) and the induction hypothesis that $g^{(m)}(0) = 0$. This completes the proof of Theorem 3.

Since (12) is locally of class C^ω, Theorem 1 guarantees the existence of an analytic stable (unstable) manifold which is unique. It is natural, therefore, to ask the following questions: Is the center manifold unique? Does there exist a C^ω center manifold? The answer to both questions is no. For a demonstration of nonuniqueness see Section 5 of this paper and Kelley [4]. That system (12) need not have an analytic center manifold is shown by the following example.

Example 1. Consider the system in the xy-plane given by

$$\dot{x} = x^2, \qquad \dot{y} = y - x^3.$$

This system has an analytic center manifold if and only if there exists a $g \in [x]_2^1$ such that $y = g(x)$ is a solution of

$$y = x^2[(dy/dx) + x].$$

Assuming that $y = g(x)$ has the power series expansion

$$g(x) = \sum_{n=3}^{\infty} a_n x^n,$$

and substituting formally, we obtain the recursion $a_3 = 1$, $a_n = (n - 1)a_{n-1}$ for $n > 3$. Therefore, $| a_n/a_{n-1} | \to \infty$ as $n \to \infty$, which implies that the radius of convergence of $g(x)$ is zero.

5. The Phase-Portrait in a Neighborhood of the Origin

We shall now describe the phase-portrait of (1) in a neighborhood of the origin in a certain geometric sense that will be developed in this section. In view of the work of Section 3, we may assume that the system under consideration is (12).

The geometric approach that we shall employ has as its point of departure the analysis of Mendelson [6]. We shall make use of the following notations and definitions: Let $\xi = (x_0, x)$, and let $\xi^0 = (x_0^0, x^0)$.

The double cone $\{\xi: | x | = m| x_0 |, m > 0\}$ will be denoted by $C(m)$. (50i)

The solid cone $\{\xi: | x | \leqslant m| x_0 |, m > 0\}$ will be denoted by $S(m)$. (50ii)

The sets $C(m) \cap \{\xi: 0 < x_0 < \alpha\}$ and $C(m) \cap \{\xi: -\alpha < x_0 < 0\}$, $\alpha > 0$, will be denoted by $C^+(m, \alpha)$ and $C^-(m, \alpha)$, respectively. (50iii)

The sets $S(m) \cap \{\xi: 0 < x_0 \leqslant \alpha\}$ and $S(m) \cap \{\xi: -\alpha \leqslant x_0 < 0\}$, $\alpha > 0$, will be denoted by $S^+(m, \alpha)$ and $S^-(m, \alpha)$, respectively. (50iv)

The sets $S(m) \cap \{\xi: x_0 = \alpha\}$ and $S(m) \cap \{\xi: x_0 = -\alpha\}$, $\alpha > 0$, will be denoted by $D^+(m, \alpha)$ and $D^-(m, \alpha)$, respectively. (50v)

Let $\phi(t, \xi^0)$ be the unique solution of (12) such that $\phi(0, \xi^0) = \xi^0$. For t fixed, $-\infty < t < \infty$, let $T^t: \xi^0 \to \xi^t$ be the map, defined on an open neighborhood of the origin U_t, which is given by $\xi^t = \phi(t, \xi^0)$. The sets,

$$\Gamma_\xi, \Gamma_\xi^+ \quad \text{and} \quad \Gamma_\xi^-,$$

are, respectively, the trajectory, the positive semitrajectory, and the negative semitrajectory through ξ. Let Ω be a region in E^{n+1} containing the origin in which system (12) is analytic. Let S be any subset of Ω. Then, we define

$$T(S) = \bigcup \{\Gamma_\xi: \xi \in S\}.$$

We shall use the definitions of (strict) egress point and (strict) ingress point given by Hartman [3, p. 37].

We shall use the notions of a *stable* [*unstable*] *fan-cone*, *simple fan-cone*, a *saddle-cone* (*of the first or second kind*), and a *simple saddle-cone* (*of the first or second kind*) for system (12). For the definitions of these sets see Mendelson [6].

The stable (unstable) manifold of (12) will be denoted by

$$M = \{\xi\colon x_0 = f(x), \mid x \mid < \delta\},$$

where $f \in [x]_2^n$ (cf. Theorem 1). The center manifold constructed in Section 4 will be denoted by

$$M^* = \{\xi\colon x = g(x_0), \mid x_0 \mid < \delta_*\},$$

where g satisfies the hypotheses of Theorem 3 on $\mid x_0 \mid < \delta_*$.

We now select a positive number α_0 satisfying the following conditions: $0 < \alpha_0 < \min(\delta, \delta_*)$; the cylinder $Z(\alpha_0) = \{\xi\colon \mid x_0 \mid, \mid x \mid \leqslant \alpha_0\}$ is contained in Ω;

$$\sup\{\max\{\mid \partial_x f \mid, \mid \partial_{x_0} g \mid\}\colon \xi \in Z(\alpha_0)\} \leqslant \tfrac{1}{2}.$$

We shall make extensive use of the following sets:

The cylinders $B(\alpha) = \{\xi\colon \mid x_0 \mid < \alpha, \mid x \mid = \alpha\}$ $(0 < \alpha < \alpha_0)$. (51i)

The solid cylinders $Z(\alpha) = \{\xi\colon \mid x_0 \mid, \mid x \mid \leqslant \alpha\}$ $(0 < \alpha < \alpha_0)$. We note that $S^+(1, \alpha) \cup S^-(1, \alpha) \subset Z(\alpha)$, and $D^+(1, \alpha)$ and $D^-(1, \alpha)$ form the upper and lower boundary, respectively, of $Z(\alpha)$. (51ii)

The sets $E(\alpha) = Z(\alpha) - [S^+(1, \alpha) \cup S^-(1, \alpha)]$ $(0 < \alpha < \alpha_0)$. (51iii)

The sets $B^+(\alpha) = B(\alpha) \cap \{\xi\colon f(x) < x_0 < \alpha\}$ and $B^-(\alpha) = B(\alpha) \cap \{\xi\colon -\alpha < x_0 < f(x)\}$ $(0 < \alpha < \alpha_0)$. (51iv)

The sets $Z^+(\alpha) = Z(\alpha) \cap \{\xi\colon f(x) < x_0 \leqslant \alpha\}$ and $Z^-(\alpha) = Z(\alpha) \cap \{\xi\colon -\alpha \leqslant x_0 < f(x)\}$ $(0 < \alpha < \alpha_0)$. (51v)

The condition $0 < \alpha < \alpha_0$ imposed in the above definitions has a number of desirable consequences: Both M and M^* are defined in an open neighborhood U of the origin in which (12) is analytic and such that $Z(\alpha) \subset U$ for all α $(0 < \alpha < \alpha_0)$. We have $M \cap \{\xi\colon \mid x \mid \leqslant \alpha\} \subset E(\alpha)$ and

$$M^* \cap \{\xi\colon \mid x_0 \mid \leqslant \alpha\} \subset \{0\} \cup S^+(1, \alpha) \cup S^-(1, \alpha)$$

for all such α. Moreover, the distance between

$$M \cap Z(\alpha) \qquad \text{and} \qquad D^+(1, \alpha) \cup D^-(1, \alpha)$$

and the distance between $M^* \cap Z(\alpha)$ and $B(\alpha)$ is not less than $\alpha/2$ for each such α. Therefore, for each $\alpha \in (0, \alpha_0)$ M^* intersects each of the disks $D^+(1, \alpha)$ and $D^-(1, \alpha)$ in a unique point while M intersects $B(\alpha)$ in an $(n - 1)$-dimensional analytic manifold $M_\alpha = M \cap B(\alpha)$. We note that M_α separates $B(\alpha)$ into two components, namely $B^+(\alpha)$ and $B^-(\alpha)$, which have M_α as their common boundary. Also $M \cap Z(\alpha)$ separates $Z(\alpha)$ into the components $Z^+(\alpha)$ and $Z^-(\alpha)$, having $M \cap Z(\alpha)$ as their common boundary. As an aid in visualizing these relationships see Fig. 1.

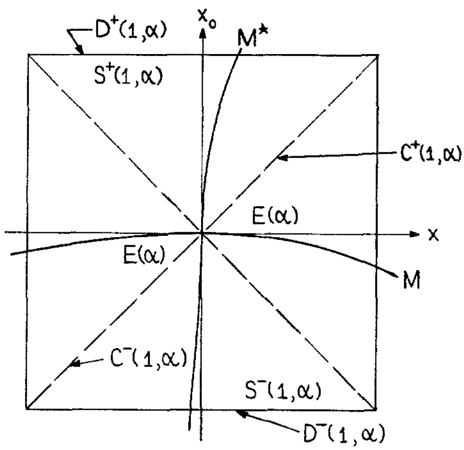

FIG. 1. $Z(\alpha)$ for $n = 1$.

The description of the local phase portrait for (12) will be given in terms of the following definitions.

The cylinder $Z^+(\alpha)$ is said to be a *stable* [*unstable*] *fan-cylinder for* (12) *if*:

(i) $\xi \in B^+(\alpha) \cup D^+(1, \alpha)$ implies that

$$\Gamma_\xi^+ \subset Z^+(\alpha)[\Gamma_\xi^- \subset Z^+(\alpha)].$$

That is, the vector field given by (12) on $B^+(\alpha) \cup D^+(1, \alpha)$ does not point out of [into] $Z^+(\alpha)$.

(ii) 0 is asymptotically stable with respect to $T(Z^+(\alpha))$ as $t \to \infty[t \to -\infty]$. (52a)

If in (52a) every $\Gamma_\xi \subset T(Z^+(\alpha))$ is tangent to the x_0-axis at the origin, then $Z^+(\alpha)$ will be called a *simple fan-cylinder*. (52b)

$Z^+(\alpha)$ is said to be a *saddle-cylinder for* (12) [*of the first kind*] if:

(i) If $\xi \in B^+(\alpha)$, then ξ is a strict ingress point of the interior of $Z^+(\alpha)$.

(ii) $\xi \in B^+(\alpha)$ implies that $\Gamma_\xi^+ \not\subset Z^+(\alpha)$.

(iii) If ξ is contained in the interior of $D^+(1, \alpha)$, then ξ is a strict egress point of the interior of $Z^+(\alpha)$.

(iv) Let $S = \{\xi \colon \xi \in D^+(1, \alpha)$ and $\Gamma_\xi^- \subset Z^+(\alpha)\}$. Then 0 is negatively asymptotically stable with respect to $T(S)$. (53a)

$Z^+(\alpha)$ is said to be a *saddle-cylinder* (*of the second kind*) *for* (12) if it is a saddle-cylinder of the first kind for (12′), where (12′) is obtained from (12) via the map $t \to -t$. (53b)

$Z^+(\alpha)$ is said to be a *simple saddle-cylinder* (of the first or second kind) for (12) if the set S of (53a, iv) consists of exactly one point. (53c)

Definitions (52a, b) and (53a, b, c) also apply to $Z^-(\alpha)$ in the obvious way.

Remark 2. If $Z^+(\alpha)$ is a simple saddle-cylinder of the first (second) kind, the unique negative semitrajectory (positive semitrajectory) which intersects $D^+(1, \alpha)$ and is contained in $Z^+(\alpha)$ must coincide with $M^* \cap Z^+(\alpha)$. A similar statement applies to $Z^-(\alpha)$.

As an aid in visualizing the above ideas, we refer the reader to Figs. 2 and 3.

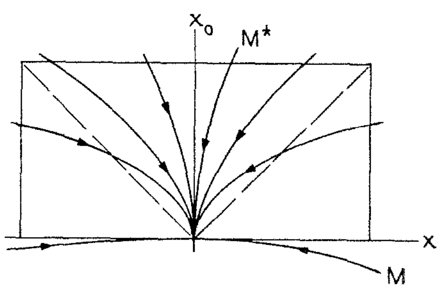

FIG. 2. $Z^+(\alpha)$ a simple, stable fan-cylinder.

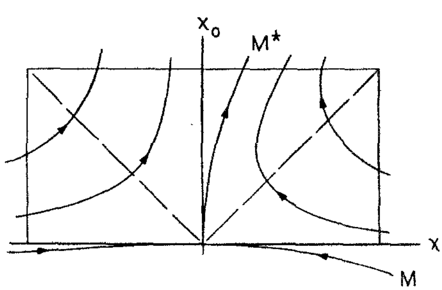

FIG. 3. $Z^+(\alpha)$ a simple saddle-cylinder of the first kind.

The following cases exhaust all possibilities for (12):

Case 1. k is an even integer $\geqslant 2$;

Case 2. k is an odd integer $\geqslant 3$.

In either case (a) $\operatorname{Re} \lambda_i < 0$, or (b) $\operatorname{Re} \lambda_i > 0$ $(i = 1,\ldots, n)$.

Referring to Mendelson [6, pp. 171–175], one sees that (12) satisfies his conditions (i)–(iv); therefore, we have the following result.

THEOREM 4 (Mendelson). *For every* $m > 0$ *there exists an* $\alpha(m)$, $0 < \alpha(m) < \alpha_0$, *such that for each* $\alpha \in (0, \alpha(m)]$ *system* (12) *has the following local behavior with respect to the cases indicated*:

Case 1a. *$S^+(m, \alpha)$ is a simple saddle-cone (of the first kind). $S^-(m, \alpha)$ is a simple stable fan-cone.*

Case 1b. *$S^+(m, \alpha)$ is an unstable, simple fan-cone. $S^-(m, \alpha)$ is a simple saddle-cone (of the second kind).*

Case 2a. *Both $S^+(m, \alpha)$ and $S^-(m, \alpha)$ are simple saddle-cones (of the first kind).*

Case 2b. *Both $S^+(m, \alpha)$ and $S^-(m, \alpha)$ are unstable simple fan-cones.*

We shall now give a complete description of the local phase-portrait of (12) in terms of the definitions (52a, b) and (53a, b, c). There is one case, however, where the literature already provides such a description: if (*) in Remark 1 (given at the end of Section 3) obtains, Theorem 3 of Mendelson [6] gives the desired result. But this will also be a consequence of the more general result that is proved in what follows.

First, we need some preliminary results. The following well known property is easily derived from (11iii) by choosing ϵ sufficiently small: There exists a $\gamma > 0$ such that in Cases 1a and 2a,

$$\langle Ax, x \rangle \leqslant -2\gamma | x |^2, \tag{54}$$

while in Cases 1b and 2b,

$$\langle Ax, x \rangle \geqslant 2\gamma | x |^2. \tag{55}$$

It follows easily from (11i), (51iii), (54), and (55) that for α sufficiently small, $\xi = (x_0, x) \in E(\alpha)$ implies that

$$d| x |/dt \leqslant -\gamma | x | \qquad \text{for Cases 1a and 2a,} \tag{56}$$

$$d| x |/dt \geqslant \gamma | x | \qquad \text{for Cases 1b and 2b;} \tag{57}$$

whence, we obtain the following lemma.

LEMMA 10. *Let $\alpha(1)$ be as in Theorem 4. Then, there exists a number α_*, $0 < \alpha_* \leqslant \alpha(1)$, such that whenever $\alpha \in (0, \alpha_*]$ every point of $B(\alpha)$ is a strict ingress (egress) point of the interior of $Z(\alpha)$ in Cases 1a and 2a (Cases 1b and 2b).*

We shall now prove the main result of this section.

THEOREM 5. *Let α_* be as in Lemma 10. Then, for every $\alpha \in (0, \alpha_*]$ the following configurations apply to (12) for the cases indicated:*

Case 1a. *$Z^+(\alpha)$ is a simple saddle-cylinder (of the first kind). $Z^-(\alpha)$ is a simple stable fan-cylinder.*

Case 1b. *$Z^+(\alpha)$ is a simple unstable fan-cylinder. $Z^-(\alpha)$ is a simple saddle-cylinder (of the second kind).*

Case 2a. *$Z^+(\alpha)$ and $Z^-(\alpha)$ are both simple saddle-cylinders (of the first kind).*

Case 2b. *Both $Z^+(\alpha)$ and $Z^-(\alpha)$ are simple unstable fan-cylinders.*

On the stable (unstable) manifold we have the following behavior: Let $\xi \in Z(\alpha)M$, and let γ be as defined previously. Then,

$$| x_0{}^t | \leqslant | x^t | \leqslant | x | e^{-\gamma t} \, (t \geqslant 0) \qquad \textit{for Cases 1a and 2a,} \tag{58}$$

and

$$| x_0{}^t | \leqslant | x^t | \leqslant | x | e^{\gamma t} \, (t \leqslant 0) \qquad \textit{for Cases 1b and 2b.} \tag{59}$$

Condition (58) [(59)] implies that in Cases 1a and 2a [Cases 1b and 2b] the origin is exponentially asymptotically stable with respect to M as $t \to \infty$ [$t \to -\infty$]. This type of stability is typical for the stable [unstable] manifold in a more general setting, cf. Hartman [6, Theorem 6.1, p. 242].

Proof of Theorem 5. The last part of the statement concerning the behavior on M is an easy consequence of (56), (57), and the fact that $Z(\alpha) \cap M \subset E(\alpha)$.

We shall now prove the decomposition into cylinders stated in the first part of Theorem 5. Denote $T^t\xi$ by $\xi^t = (x_0{}^t, x^t)$. Let $\alpha \in (0, \alpha_*]$, and recall that

$$\alpha_* \leqslant \alpha(1). \tag{60}$$

By (60), if $\xi \in S^+(1, \alpha)$ [$\xi \in S^-(1, \alpha)$] the geometry of $\Gamma_\xi \cap S^+(1, \alpha)$ $\Gamma_\xi \cap S^-(1, \alpha)$] is given by Theorem 4. Therefore, it is necessary only to consider points in the sets $Z^+(\alpha) \cap E(\alpha)$ and $Z^-(\alpha) \cap E(\alpha)$ which we will denote by $E^+(\alpha)$ and $E^-(\alpha)$, respectively.

Consider Cases 1a and 2a, and suppose $\xi \in E^+(\alpha)$ [$\xi \in E^-(\alpha)$]. In view of (60), the invariance of M, Theorem 4, and Lemma 10, it follows that either $\Gamma_\xi{}^+ \subset E^+(\alpha)$ [$\Gamma_\xi{}^+ \subset E^-(\alpha)$], or

$$(H_+) \; \xi^t \in S^+(1, \alpha)[\xi^t \in S^-(1, \alpha)] \qquad \text{for some} \quad t > 0.$$

Similarly, in Cases 1b and 2b, $\xi \in E^+(\alpha)$ $[\xi \in E^-(\alpha)]$ implies that either $\Gamma_\xi^- \subset E^+(\alpha)$ $[\Gamma_\xi^- \subset E^-(\alpha)]$, or

$$(H_-)\ \xi^t \in S^+(1, \alpha)[\xi^t \in S^-(1, \alpha)] \qquad \text{for some} \quad t < 0.$$

By (60), Lemma 10, the definitions of the sets involved, and Theorem 4, the proof will be complete if we show that in Cases 1a and 2a (H_+) holds, and in Cases 1b and 2b (H_-) holds for each $\xi \in E^+(\alpha)$ $[\xi \in E^-(\alpha)]$.

Consider Case 1a or 2a. Assume, on the contrary, that there exists a point $\xi \in E^+(\alpha)$ $[\xi \in E^-(\alpha)]$ for which (H_+) is false. Then $\Gamma_\xi^+ \subset E^+(\alpha)$ $[\Gamma_\xi^+ \subset E^-(\alpha)]$, which implies, in particular, that ξ^t is defined for all $t \geqslant 0$,

$$\Gamma_\xi^+ \cap [M \cup M^*] \text{ is empty,} \tag{61}$$

$|x^t| > 0$ for all $t \geqslant 0$, and

$$\limsup_{t\to\infty}(|x_0{}^t|/|x^t|) \leqslant 1. \tag{62}$$

By (56), $|x^t| \to 0$ as $t \to \infty$, and, therefore, (62) implies that

$$\xi^t \to 0 \qquad \text{as} \qquad t \to \infty. \tag{63}$$

Let the map R_1: $(x_0, x) \to (u_0, u)$ be defined by $u_0 = x_0 - f(x)$, $u = x$, where f is the defining function of the stable (unstable) manifold M, and let R_2: $(u_0, u) \to (w_0, w)$ denote the map $w_0 = u_0$, $w = u - \tilde{g}(u_0)$ where $\tilde{g}$ is the defining function of the center manifold $R_1(M^*)$ in the (u_0, u)-system. The composite $R = R_2R_1$ is given by

$$R\colon w_0 = x_0 - f(x), \qquad w = x - G(\xi),$$

where $G(\xi) = \tilde{g}(x_0 - f(x))$. Let $\eta = (w_0, w)$. The inverse of R is

$$R^{-1}\colon x_0 = w_0 + F(\eta), \qquad x = w + \tilde{g}(w_0),$$

where $F(\eta) = f(w + \tilde{g}(w_0))$. Therefore, R is a local C^∞-diffeomorphism, defined on an open neighborhood U of the origin, which transforms (12) into

$$\dot{w}_0 = W_0(\eta), \qquad \dot{w} = Aw + W(\eta), \tag{64}$$

where W_0, W are of class C^∞ on U and vanish at $\eta = 0$; $\partial_\eta W_0$, $\partial_\eta W \equiv 0$ at $\eta = 0$; $W_0(0, w) \equiv 0$ for all $(0, w) \in R(U)$; $W(w_0, 0) \equiv 0$ for all $(w_0, 0) \in R(U)$.

We may assume $\Gamma_\xi^+ \subset U$. Then $R(\xi^t)$, which we shall denote by

$\eta^t = (w_0{}^t, w^t)$, is the solution of (64) passing through $R(\xi) = \eta$ at $t = 0$. By virtue of the properties of (64), it follows from (61)–(63) that

$$| w_0{}^t |, | w^t | > 0 \qquad \text{for all} \quad t \geqslant 0, \tag{65}$$

$$\eta^t \to 0 \qquad \text{as} \qquad t \to \infty, \tag{66}$$

and

$$\limsup_{t\to\infty}(| w_0{}^t |/| w^t |) \leqslant 1. \tag{67}$$

For the remainder of the argument we shall use the discussion following Theorem 3. For each $\mu > 0$, define R_μ: $\eta \to (y_0, y)$ by $(y_0, y) = \mu^{-1}\eta$. Let $\zeta = (y_0, y)$. Then for sufficiently small μ, the map R_μ, when restricted to some open neighborhood U_μ of the origin, defines a C^ω-diffeomorphism taking (64) into (28). In view of the local invariance of both $w_0 = 0$ and $w = 0$, we may assume

$$Y_0(0, y, \mu) \equiv 0, \qquad Y(y_0, 0, \mu) \equiv 0, \tag{68}$$

for all $\mu \in (0, \mu_0)$.

By (66), we may assume that $\Gamma_\eta^+ \subset U_\mu$ for any $\mu \in (0, \mu_0)$. Let $R_\mu(\eta^t) = \zeta^t = (y_0{}^t, y^t)$. It follows easily from (65), (66), (67), and the definition of R_μ that for each $\mu \in (0, \mu_0)$ the following properties obtain: $| y_0{}^t |, | y^t | > 0$ for all $t \geqslant 0$, $\zeta^t \to 0$ as $t \to \infty$, and

$$\limsup_{t\to\infty}(| y_0{}^t |/| y^t |) \leqslant 1. \tag{69}$$

The group of maps $\{T_\mu{}^t\}$ associated with (28) is given by (29); it satisfies Lemma 2. Moreover, by virtue of (68) this group also satisfies an additional property: $V_0(t, 0, y, \mu) \equiv 0$, $V(t, y_0, 0, \mu) \equiv 0$ for all $-\infty < t < \infty$, and each $\mu \in (0, \mu_0)$. Consequently, the map $T_\mu = T_\mu{}^1$ given by (31) is such that

$$V_0(0, y; \mu) \equiv 0, \qquad V(y_0, 0; \mu) \equiv 0 \qquad \text{for each} \quad \mu \in (0, \mu_0). \tag{70}$$

Since we are considering Case 1a or 2a, (32) is replaced by

$$| C | = c < 1. \tag{71}$$

It follows from Lemma 2, (70), and the mean value theorem that

$$| V_0(\zeta; \mu)| \leqslant \theta_1(\mu)| y_0 |, \qquad \text{and} \qquad | V(\zeta; \mu)| \leqslant \theta_1(\mu)| y |, \tag{72}$$

where $\theta_1(\mu)$ is as in (30iv). For each $n \in N$, define $\zeta^n = T_\mu{}^n(\zeta)$, where $\zeta = \zeta^0$. Since $y_0^{n+1} = y_0{}^n + V(\zeta^n; \mu)$ and $y^{n+1} = Cy^n + V(\zeta^n; \mu)$, it follows from (71) and (72) that for each $n \in N$,

$$| y_0^{n+1} | \geqslant [1 - \theta_1(\mu)] \, | y_0{}^n |, \qquad \text{and} \qquad | y^{n+1} | \leqslant [c + \theta_1(\mu)] \, | y^n |;$$

whence, by induction on n we find that for each $n \in N$,

$$|y_0^n| \geqslant [1 - \theta_1(\mu)]^n |y_0|, \qquad \text{and} \qquad |y^n| \leqslant [c + \theta_1(\mu)]^n |y| \quad (73)$$

for each $\mu \in (0, \mu')$, where $\mu' \leqslant \mu_0$ is such that $\theta_1(\mu) < 1$ whenever $0 < \mu < \mu'$. Now choose $\mu \in (0, \mu')$ so small that $\theta_1(\mu) < (1 - c)/2$. Then, by (73) and the definition of μ,

$$(|y_0^n|/|y^n|) \geqslant K^n(|y_0|/|y|) \qquad \text{for all} \quad n \in N,$$

where $K > 1$. Since $(|y_0|/|y|) > 0$, this contradicts (5.20). Consequently, (H_+) holds for each ξ in $E^+(\alpha)$ or $E^-(\alpha)$.

To show that (H_-) holds in Case 1b or 2b, one need only make the change of variables $t \to -t$ in (12) and then repeat the foregoing argument. This completes the proof of Theorem 5.

It follows from Theorem 5 that the origin is unstable for (12) in Cases 1a, 1b and 2a; in Case 2b the origin is negatively asymptotically stable. Since (1) is locally C^ω-equivalent to (12), the following result due to Liapunov [5, p. 301] is a corollary of Theorem 5.

COROLLARY 1. *The origin is either unstable, positively asymptotically stable, or negatively asymptotically stable with respect to system* (1).

When n (the dimension of x) $= 1$, Theorem 5 yields the celebrated theorem of Bendixson [1, pp. 45–58].

COROLLARY 2. *Let* $n = 1$. *The phase portrait of system* (1) *about the origin is one of the following types: node, saddle-point, two hyperbolic sectors and a fan.*

If Γ_1 is a trajectory approaching the origin tangent to the x_0-axis from above the stable (unstable) manifold M, and Γ_2 is a trajectory approaching the origin tangent to the x_0-axis from below M, then $\Gamma_1 \cup \{0\} \cup \Gamma_2$ is (locally) a C^k center manifold for (12). Thus we also obtain the following result.

COROLLARY 3. *In Case 2a system* (12) *has a unique center manifold* (*of class* C^∞). *In all other cases the set of all* C^k *center manifolds of* (12) *has cardinal number* $2^{\aleph_0}$.

The referee has informed me that the uniqueness of the center manifold in Case (2a) follows from Kelley [8, p. 337].

Consider the system obtained from (12) by retaining only the lowest order terms in the equations, namely,

$$\dot{u}_0 = u_0^k, \qquad \dot{u} = Au. \qquad (74)$$

The solution of (74) passing through (u_0^0, u^0) at $t = 0$ is

$$u_0(t) = u_0^0[1 + (1 - k)(u_0^0)^{k-1}t]^{1/1-k}, \qquad u(t) = e^{tA}u^0.$$

Therefore, (74) satisfies Theorem 5. In view of the geometric similarity of (12) and (74) it is natural to ask if there is a local diffeomorphism (or at least a homeomorphism) taking (12) into (74). Regarding this equation the author has obtained the following result: assuming (*) of Remark 1 at the end of Section 3, together with a certain mild restriction, there exists a local C^∞-diffeomorphism which takes (12) into (74).

Acknowledgments

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